

A SHORT GEOMETRIC PROOF OF THE ZALCMAN AND BIEBERBACH CONJECTURES

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ABSTRACT. We show that complex geometric features of Teichmüller spaces create explicitly the extremals of generic homogeneous holomorphic functionals on univalent functions. In particular this gives proofs of the well-known Zalcman and Bieberbach conjectures and many new distortion theorems.

2010 Mathematics Subject Classification: Primary: 30C50, 30C75, 30F60, 32Q45; Secondary 30C55, 30C62

Key words and phrases: Univalent function, homogeneous functional, Teichmüller space, Bieberbach conjecture, Zalcman's conjecture, quasiconformal map, invariant metrics, complex geodesic

1. INTRODUCTION

Our aim is to show that complex geometry of the universal Teichmüller space and Teichmüller space of the punctured disk describes explicitly the extremals of generic homogeneous holomorphic functionals on univalent functions. This yields, in particular, the proof of the famous Zalcman and Bieberbach conjectures and many new distortion theorems.

1.1. Classes of functions and general homogeneous holomorphic functionals. The holomorphic functionals on the classes of univalent functions depending on the Taylor coefficients of these functions play an important role in various geometric and physical applications of complex analysis, for example, in view of their connection with string theory and with a holomorphic extension of the Virasoro algebra. These coefficients reflect the fundamental intrinsic features of conformal maps. Thus estimating them still remains an important problem in geometric function theory.

We consider the univalent functions on the unit disk $\Delta = \{|z| < 1\}$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

These functions form the well-known class S . Their inversions $F_f(z) = 1/f(1/z)$ form the collection Σ of univalent nonvanishing functions ($F_f(z) \neq 0$) on the complementary disk $\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$ with expansions

$$F_f(z) = 1/f(1/z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots \quad (1.1)$$

Date: August 11, 2014 (shortZB.tex).

Easy computations yield that the coefficients a_n and b_j are related by

$$b_0 + a_2 = 0, \quad b_n + \sum_{j=1}^n b_{n-j} a_{j+1} + a_{n+2} = 0, \quad n = 1, 2, \dots, \quad (1.2)$$

which implies successively the representations of a_n by b_j . One gets

$$a_n = (-1)^{n-1} b_0^{n-1} - (-1)^{n-1} (n-2) b_1 b_0^{n-3} + \text{lower terms with respect to } b_0; \quad (1.3)$$

in particular,

$$\begin{aligned} a_2 &= -b_0, \quad a_3 = -b_1 + b_0^2, \quad a_4 = -b_2 + 2b_1 b_0 - b_0^3, \\ a_5 &= -b_3 + 2b_2 b_0 + b_1^2 - 3b_1 b_0^2 + b_0^4, \\ a_6 &= -b_4 + 2b_3 b_0 + 2b_2 b_1 - 3b_2 b_0^2 - 3b_1^2 b_0 + 4b_1 b_0^3 - b_0^5, \\ a_7 &= b_0^6 - 5b_1 b_0^4 - b_1^3 + 4b_2 b_0^3 + b_2^2 + (6b_1^2 - 3b_3) b_0^2 \\ &\quad + 2b_1 b_3 + (-6b_1 b_2 + 2b_4) b_0 - b_5, \dots \end{aligned}$$

We shall essentially use this connection.

Consider a general holomorphic distortion functional on S of the form

$$J(f) = J(a_2, \dots, a_n; (f^{(\alpha_1)}(z_1)); \dots; (f^{(\alpha_p)}(z_p))), \quad (1.4)$$

where z_1, \dots, z_p are the distinct fixed points in $\Delta \setminus \{0\}$ with assigned orders m_1, \dots, m_p , respectively, $(f^{(\alpha_1)}(z_1)) = f''(z_1), \dots, f^{(m_1)}(z_1)$; $(f^{(\alpha_p)}(z_p)) = f''(z_p), \dots, f^{(m_p)}(z_p)$. Assume that J is a polynomial in all of its variables.

Substituting the expressions of a_j by b_m from (1.2) and calculating $f^{(q)}(z_j)$ in terms of F_f , one obtains a polynomial $\tilde{J}(F)$ of the Taylor coefficients b_0, b_1, \dots, b_{n-2} and of the corresponding derivatives $F_f^{(q)}(\zeta_j)$ at the points $\zeta_j = 1/z_j \in \Delta^* \setminus \{\infty\}$, regarded as a representation of $J(f)$ on the class Σ . Here $q = 2, \dots, m_j$, $j = 1, \dots, p$.

Assume that the functional (1.4) is homogeneous with a degree $d = d(J)$ (depending on n and m_1, \dots, m_p) with respect to the homotopy

$$f(z, t) = t^{-1} f(tz) = z + a_2 t + a_3 t^2 + \dots : \Delta \times \overline{\Delta} \rightarrow \mathbb{C}$$

such that $f(z, 0) \equiv z$, $f(z, 1) = f(z)$ so that

$$J(f_t) = t^d J(f).$$

This homotopy is a special case of holomorphic motions with complex parameter t running over the disk Δ . The functional $\tilde{J}(F)$ on Σ admits a similar homogeneity.

The existence of extremal functions of $J(f)$ and $\tilde{J}(F)$ follows from compactness of both classes S and Σ in the topology of locally uniform convergence on Δ and Δ^* , respectively.

1.2. The Bieberbach and Zalcman conjectures. There were several classical conjectures about the coefficients. They include the Bieberbach conjecture that in the class S the coefficients are estimated by $|a_n| \leq n$, as well as several other well-known conjectures that imply the Bieberbach conjecture. Most of them have been proved by the de Branges theorem [DB].

In the 1960s, Lawrence Zalcman posed the conjecture that *for any $f \in S$ and all $n \geq 3$,*

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad (1.5)$$

with equality only for the Koebe function

$$\kappa_\theta(z) = \frac{z}{(1 - e^{i\theta}z)^2} = z + \sum_2^\infty n e^{-i(n-1)\theta} z^n, \quad 0 \leq \theta \leq 2\pi, \quad (1.6)$$

which maps the unit disk onto the complement of the ray

$$w = -te^{-i\theta}, \quad \frac{1}{4} \leq t \leq \infty.$$

This remarkable conjecture also implies the Bieberbach conjecture and remained an intriguing very difficult open problem for all $n > 6$.

The original aim of Zalcman's conjecture was to prove the Bieberbach conjecture using the famous Hayman theorem on the asymptotic growth of coefficients of individual functions, which states that *for each $f \in S$, we have the inequality*

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{n} = \alpha \leq 1,$$

with equality only when $f = \kappa_\theta$; here $\alpha = \lim_{r \rightarrow 1} (1 - r)^2 \max_{|z|=r} |f(z)|$ (see [Ha]).

Indeed, assuming that n is sufficiently large and estimating a_{2n-1} in (1.5) by $|a_{2n-1}| \leq 2n - 1$, one obtains

$$|a_n|^2 \leq (n - 1)^2 + |a_{2n-1}| \leq (n - 1)^2 + 2n - 1 = n^2,$$

which proves the Bieberbach conjecture for this n , and successively for all preceding coefficients.

It was realized almost immediately that the Zalcman conjecture implies the Bieberbach conjecture, and in a very simple fashion, without Hayman's result and without other prior results from the theory of univalent functions.

Note that the case $n = 2$ is rather simple and somewhat exceptional. The inequality $|a_2^2 - a_3| \leq 1$ is well known, but in this case there are two extremal functions of different kinds: the Koebe function $\kappa_\theta(z)$ and the odd function

$$\kappa_{2,\theta}(z) := \sqrt{\kappa_\theta(z^2)} = \sum_{n=0}^\infty e^{in\theta} z^{2n+1}. \quad (1.7)$$

The estimate (1.1) was established for $n \leq 6$ in [Kr4], [Kr7] (for $n = 4, 5, 6$ without uniqueness of the extremal function). In [BT], [Ma], this conjecture was proved for certain special subclasses of S .

2. MAIN THEOREMS

2.1. General theorem. It is well known that the Koebe function κ_θ is extremal for many variational problems in the theory of conformal maps (accordingly, its root transforms

$$\kappa_{m,\theta}(z) = \kappa_\theta(z^m)^{1/m} = \frac{z}{(1 - e^{i\theta}z^m)^{2/m}} = z + \frac{2e^{i\theta}}{m} z^{m+1} + \frac{m-2}{m^2} z^{2m+1} + \dots, \quad m = 2, 3, \dots, \quad (2.1)$$

are extremal among the maps with symmetries).

Our first main theorem sheds new light on this phenomenon and provides a large class of functionals maximized by these functions.

Theorem 2.1. *Let $J(f)$ be a homogeneous polynomial functional on S of the form (2.4) whose representation $\tilde{J}(F_f)$ in the class Σ does not contain free terms $c_d b_0^d$ but contains nonzero terms with the coefficient b_1 of inversions F_f . Then for all $f \in S$, we have the sharp bound*

$$|J(f)| \leq \max_m |J(\kappa_{m,\theta})|, \quad (2.2)$$

and this maximum is attained on some $\kappa_{m_0,\theta}$ ($m_0 \geq 1$). If J has an extremal with

$$b_1 = a_2^2 - a_3 \neq 0, \quad (2.3)$$

then $|b_1| = 1$ and

$$|J(f)| \leq \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\}. \quad (2.4)$$

The assumption (2.3) is equivalent to

$$S_f(0) = - \lim_{z \rightarrow \infty} z^4 S_{F_f}(z) \neq 0,$$

where S_f denotes the **Schwarzian derivative** of f in Δ defined by

$$S_f = (f''/f')' - (f''/f')^2/2.$$

The examples of some well-known functionals, for example, $J(f) = a_2^2 - \alpha a_3$ with $0 < \alpha < 1$ and $J(F_f) = b_m$ ($m > 1$), show that the assumptions on the initial coefficients b_0 and b_1 cannot be omitted.

2.2. Applications. The Zalcman functional

$$J_n(f) = a_n^2 - a_{2n-1}$$

is a special case of (2.4) with homogeneity degree $2n - 2$. For this functional, we obtain from Theorem 2.1 a complete result proving the Zalcman conjecture.

Theorem 2.2. *For all $f \in S$ and any $n \geq 3$, we have the sharp estimate (1.5), with equality only for $f = \kappa_\theta$.*

As a consequence, one obtains also a new proof of the Bieberbach conjecture.

Theorem 2.1 also provides other new distortion theorems concerning the higher coefficients. These results are presented in Section 6. In the last section, we show how the proof of Theorem 2.1 yields asymptotic estimating the growth rate of generic homogeneous functionals on an individual function with quasiconformal extension.

2.3. Connection with geometry of Teichmüller spaces. Our approach to these problems is geometric. Its origins go back to [Kr7] where the proof of Zalcman's conjecture for the initial coefficients was given.

It suffices to find the bound of J on functions f admitting quasiconformal extensions across the unit circle and close this set in weak topology determined by locally uniform convergence on Δ . Denote the subset of such f by S^0 and the set of corresponding $F_f \in \Sigma$ by Σ^0 .

Such functions are naturally connected with the universal Teichmüller space $\mathbf{T} = \mathbf{T}(\Delta)$ and the Teichmüller space $\mathbf{T}_1 = \mathbf{T}(\Delta \setminus \{0\})$ of the punctured disk. Accordingly, the original functional $J(f)$ is lifted to a holomorphic functional on \mathbf{T}_1 , and its sharp upper bound is obtained using deep geometric features of this space. Application of metrics of negative generalized curvature in the lines of [Kr7] allows us to estimate the functional from below giving the same asymptotic bound.

In fact, we establish that every homogeneous holomorphic functional on S satisfying the assumptions of Theorem 2.1 determines a complex geodesic in the space \mathbf{T}_1 generated by some $\kappa_{m,\theta}$.

3. BACKGROUND

We briefly present here certain results underlying the proof of the key Theorem 2.1. The exposition is adapted to our special cases.

3.1. A glimpse at complex geometry of Teichmüller spaces \mathbf{T} and \mathbf{T}_1 . (a) First recall that the universal Teichmüller space \mathbf{T} is the space of quasiconformal homeomorphisms of the unit circle $S^1 = \partial\Delta$ factorized by Möbius maps. All Teichmüller spaces have their isometric copies in \mathbf{T} .

The canonical complex Banach structure on \mathbf{T} is defined by factorization of the ball of the **Beltrami coefficients** (or complex dilatations)

$$\mathbf{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu|_{\Delta^*} = 0, \|\mu\| < 1\}, \quad (3.1)$$

letting $\mu_1, \mu_2 \in \mathbf{Belt}(\Delta)_1$ be equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions to the Beltrami equation $\partial_{\bar{z}}w = \mu\partial_zw$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $S^1 = \partial\Delta^*$ (hence, on $\overline{\Delta^*}$). The equivalence classes $[w^\mu]$ are in one-to-one correspondence with the Schwarzian derivatives S_w of $w = F^\mu$ on Δ^* .

The smallest dilatation $k(w) = \inf \|\mu_w\|_\infty$ among quasiconformal extensions of univalent $w|_{\Delta^*} \in \Sigma^0$ onto $\widehat{\mathbb{C}}$ is called the **Teichmüller norm** of w .

Note that for each locally univalent function $w(z)$ on a simply connected hyperbolic domain $D \subset \widehat{\mathbb{C}}$, its Schwarzian derivative S_w belongs to the complex Banach space $\mathbf{B}(D)$ of hyperbolically bounded holomorphic functions on D with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_D \lambda_D^{-2}(z) |\varphi(z)|,$$

where $\lambda_D(z)|dz|$ is the hyperbolic metric on D of Gaussian curvature -4 ; hence $\varphi(z) = O(z^{-4})$ as $z \rightarrow \infty$ if $\infty \in D$. In particular, for $D = \Delta$,

$$\lambda_\Delta(z) = 1/(1 - |z|^2). \quad (3.2)$$

The space $\mathbf{B}(D)$ is dual to the Bergman space $A_1(D)$, a subspace of $L_1(D)$ formed by integrable holomorphic functions on D .

The Schwarzians $S_{w^\mu}(z)$ with $\mu \in \mathbf{Belt}(\Delta)_1$ range over a bounded domain in the space $\mathbf{B} = \mathbf{B}(\Delta^*)$. This domain models the universal Teichmüller space \mathbf{T} , and the factorizing projection

$$\phi_{\mathbf{T}}(\mu) = S_{w^\mu} : \mathbf{Belt}(\Delta)_1 \rightarrow \mathbf{T}$$

is a holomorphic map from $L_\infty(\Delta)$ to \mathbf{B} . This map is a split submersion, which means that $\phi_{\mathbf{T}}$ has local holomorphic sections (see, e.g., [GL]).

Both equations $S_w = \varphi$ and $\partial_{\bar{z}}w = \mu\partial_zw$ (on Δ^* and Δ , respectively) determine their solutions in Σ^0 up to translations $w \mapsto w + b_0$. To determine a solution w^μ uniquely, we add the condition $w^\mu(0) = 0$ going over from w^μ to the maps

$$w_1^\mu(z) = w^\mu(z) - w^\mu(0) = z - \frac{1}{\pi} \iint_{\Delta} \frac{\partial w^\mu}{\partial \bar{\zeta}} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\xi d\eta \quad (\zeta = \xi + i\eta).$$

Then the values $w^\mu(z_0)$ (for any fixed $z_0 \in \mathbb{C}$) and the Taylor coefficients b_1, b_2, \dots of $w^\mu \in \Sigma^0$ depend holomorphically on $\mu \in \mathbf{Belt}(\Delta)_1$ and on $S_{w^\mu} \in \mathbf{T}$.

The points of Teichmüller space $\mathbf{T}_1 = \mathbf{T}(\Delta^0)$ of the punctured disk $\Delta^0 = \Delta \setminus \{0\}$ are the equivalence classes of Beltrami coefficients $\mu \in \mathbf{Belt}(\Delta)_1$ so that the corresponding quasiconformal automorphisms w^μ of the unit disk coincide on both boundary components (unit circle $S^1 = \{|z| = 1\}$ and the puncture $z = 0$) and are homotopic on $\Delta \setminus \{0\}$. This space can be endowed with a canonical complex structure of a complex Banach manifold and embedded into \mathbf{T} using uniformization.

Namely, the disk Δ^0 is conformally equivalent to the factor Δ/Γ , where Γ is a cyclic parabolic Fuchsian group acting discontinuously on Δ and Δ^* . The functions $\mu \in L_\infty(\Delta)$ are lifted to Δ as the Beltrami $(-1, 1)$ -measurable forms $\tilde{\mu}d\bar{z}/dz$ in Δ with respect to Γ , i.e., via $(\tilde{\mu} \circ \gamma)\bar{\gamma}'/\gamma' = \tilde{\mu}$, $\gamma \in \Gamma$, forming the Banach space $L_\infty(\Delta, \Gamma)$.

We extend these $\tilde{\mu}$ by zero to Δ^* and consider the unit ball $\mathbf{Belt}(\Delta, \Gamma)_1$ of $L_\infty(\Delta, \Gamma)$. Then the corresponding Schwarzians $S_{w^\mu|_{\Delta^*}}$ belong to \mathbf{T} . Moreover, \mathbf{T}_1 is canonically isomorphic to the subspace $\mathbf{T}(\Gamma) = \mathbf{T} \cap \mathbf{B}(\Gamma)$, where $\mathbf{B}(\Gamma)$ consists of elements $\varphi \in \mathbf{B}$ satisfying $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ in Δ^* for all $\gamma \in \Gamma$. Most of the results about the universal Teichmüller space presented in Section 1 extend straightforwardly to \mathbf{T}_1 .

Due to the Bers isomorphism theorem, the space \mathbf{T}_1 is biholomorphically equivalent to the Bers fiber space

$$\mathcal{F}(\mathbf{T}) = \{\phi_{\mathbf{T}}(\mu), z\} \in \mathbf{T} \times \mathbb{C} : \mu \in \mathbf{Belt}(\Delta)_1, z \in w^\mu(\Delta)\}$$

over the universal Teichmüller space with holomorphic projection $\pi(\psi, z) = \psi$ (see [Be2]). This fiber space is a bounded domain in $\mathbf{B} \times \mathbb{C}$.

We shall denote the equivalence classes of $\mu \in \mathbf{Belt}(\Delta)_1$ in \mathbf{T} and \mathbf{T}_1 by $[\mu]$ and $[\mu]_1$ (also by $[w^\mu]$ and $[w^\mu]_1$), respectively.

Let $\tilde{\mathbf{T}}$ denote one of the spaces \mathbf{T} , \mathbf{T}_1 . It is a complex Banach manifold, thus it possesses the invariant Carathéodory and Kobayashi distances (the smallest and the largest among all holomorphically non-expanding metrics). Denote these metrics by $c_{\tilde{\mathbf{T}}}$ and $d_{\tilde{\mathbf{T}}}$, and let $\tau_{\tilde{\mathbf{T}}}$ be the intrinsic Teichmüller metric of this space canonically determined by quasiconformal maps. The corresponding differential (infinitesimal) Finsler forms of these metrics are defined on the tangent bundle $\mathcal{T}\tilde{\mathbf{T}}$ of $\tilde{\mathbf{T}}$. Then $c_{\tilde{\mathbf{T}}}(\cdot, \cdot) \leq d_{\tilde{\mathbf{T}}}(\cdot, \cdot) \leq \tau_{\tilde{\mathbf{T}}}(\cdot, \cdot)$, and by the Royden-Gardiner theorem the Kobayashi and Teichmüller metrics (and their infinitesimal forms) are equal, see, e.g. [EKK], [EM], [GL], [Ro1].

(b) For the spaces \mathbf{T} and \mathbf{T}_1 , there is a much stronger result established in [Kr6], [Kr9].

Theorem A. *The Carathéodory metric of the space $\tilde{\mathbf{T}}$ coincides with its Kobayashi metric, hence all invariant non-expanding metrics on $\tilde{\mathbf{T}}$ are equal its Teichmüller metric, and*

$$c_{\tilde{\mathbf{T}}}(\varphi, \psi) = d_{\tilde{\mathbf{T}}}(\varphi, \psi) = \tau_{\tilde{\mathbf{T}}}(\varphi, \psi) = \inf\{d_\Delta(h^{-1}(\varphi), h^{-1}(\psi)) : h \in \text{Hol}(\Delta, \tilde{\mathbf{T}})\}, \quad (3.3)$$

where d_Δ denotes the hyperbolic metric of the unit disk of curvature -4 (i.e., with the differential form (3.2)).

Similarly, the infinitesimal forms of these metrics coincide with the Finsler form of $\tau_{\tilde{\mathbf{T}}}$ and have holomorphic sectional curvature -4 .

Such a theorem has been proved in [Kr6] for the universal Teichmüller space. This proof is complicated and involves the technique of the Grunsky coefficient inequalities. A much simpler proof was given recently in [Kr9], and the same arguments work for $\tilde{\mathbf{T}} = \mathbf{T}_1$.

Theorem A is one of the main ingredients in the proof of our main theorems. It also has many other applications. In particular, the Teichmüller extremal disks are simultaneously geodesic for all non-expanding invariant metrics (cf. [EKK], [Ve]).

Combining Theorem A with Golusin's improvement of Schwarz's lemma, one derives the following sharp estimate of the growth of holomorphic maps on geodesic disks.

Proposition 3.1. [Kr9] *If the restriction of a holomorphic map $h : \tilde{\mathbf{T}} \rightarrow \Delta$ onto a geodesic disk $\Delta(\mu_0) = \{\phi_{\tilde{\mathbf{T}}}(t\mu_0/\|\mu_0\|_\infty) : |t| < 1\}$ has at the origin zero of order $m \geq 1$, i.e.,*

$$h_{\mu_0}(t) := h \circ \phi_{\tilde{\mathbf{T}}}(t\mu_0/\|\mu_0\|_\infty) = c_m t^m + c_{m+1} t^{m+1} + \dots,$$

then the growth of $|h|$ on this disk is estimated by

$$\begin{aligned} |h_{\mu_0}(t)| &\leq |t|^m (|t| + |c_m|) / (1 + |c_m||t|) \\ &= \tanh d_{\tilde{\mathbf{T}}} \left(\mathbf{0}, \phi_{\tilde{\mathbf{T}}} \left(|t|^m \frac{|t| + |c_m|}{1 + |c_m||t|} \frac{\mu_0}{\|\mu_0\|_\infty} \right) \right) \leq \tanh d_{\tilde{\mathbf{T}}} \left(\mathbf{0}, \phi_{\tilde{\mathbf{T}}} \left(t^m \frac{\mu_0}{\|\mu_0\|_\infty} \right) \right). \end{aligned} \quad (3.4)$$

The equality in the right inequality occurs (even for one $t_0 \neq 0$) only when $|c_p| = 1$; then $h_{\mu_0}(t)$ is a hyperbolic isometry of the unit disk and all terms in (3.4) are equal.

Golusin's version mentioned above asserts that a holomorphic function

$$g(t) = c_m t^m + c_{m+1} t^{m+1} + \dots : \Delta \rightarrow \Delta \quad (c_m \neq 0, \quad m \geq 1)$$

is estimated in Δ by

$$|g(t)| \leq |t|^m \frac{|t| + |c_m|}{1 + |c_m||t|},$$

and the equality occurs only for $g_0(t) = t^m(t + c_m)/(1 + \bar{c}_m t)$ (see [Go, Ch. 8]).

On the other hand, it follows from Theorem A and weak* compactness of the closure of $\tilde{\mathbf{T}}$ in $\mathbf{B} \times \Delta$ that for any fixed $t_0 \neq 0$ there is a holomorphic map $j(\varphi) : \tilde{\mathbf{T}} \rightarrow \Delta$ such that

$$d_\Delta(0, j \circ \phi_{\tilde{\mathbf{T}}}(t_0 \mu_0^*)) = c_{\tilde{\mathbf{T}}}(\mathbf{0}, \phi_{\tilde{\mathbf{T}}}(t_0 \mu_0^*)) = d_{\tilde{\mathbf{T}}}(\mathbf{0}, \phi_{\tilde{\mathbf{T}}}(t_0 \mu_0^*)).$$

where $\mu_0^* = \mu_0/\|\mu_0\|_\infty$. Thus, letting

$$\eta(t) = |t|^m (|t| + |c_m|) / (1 + |c_m||t|) \leq |t|,$$

one derives

$$|h_{\mu_0}(t_0)| \leq j \circ \phi_{\tilde{\mathbf{T}}}(\eta(t_0) \mu_0^*) = \tanh d_{\tilde{\mathbf{T}}}(\mathbf{0}, \phi_{\tilde{\mathbf{T}}}(\eta(t_0) \mu_0^*)) \leq \tanh d_{\tilde{\mathbf{T}}}(\mathbf{0}, \phi_{\tilde{\mathbf{T}}}(|t_0| \mu_0^*))$$

which implies (3.4) (for details see [Kr9]).

There is also a differential analog of the inequalities (3.4) which involves the infinitesimal metrics $\mathcal{C}_{\tilde{\mathbf{T}}}$ and $\mathcal{K}_{\tilde{\mathbf{T}}}$. It will not be used here.

3.2. A holomorphic homotopy of univalent function. Similar to the functions in S , we define for each $F \in \Sigma$ with expansion (1.1) the complex homotopy

$$F_t(z) = tF\left(\frac{z}{t}\right) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \dots : \Delta^* \times \Delta \rightarrow \hat{\mathbb{C}} \quad (3.5)$$

so that $F_0(z) \equiv z$. Then $S_{F_t}(z) = t^{-2} S_F(t^{-1} z)$, and moreover, this point-wise map determines a holomorphic map

$$h_F(t) = S_{F_t}(\cdot) : \Delta \rightarrow \mathbf{B} \quad (3.6)$$

(see, e.g. [Kr3]). This map generates the **homotopy disks** $\Delta(S_F) = h_F(\Delta)$ of F in the space \mathbf{T} and its covers in \mathbf{T}_1 defined by

$$\Delta_1(S_F, b_0) := \{(S_{F_t}, tb_0) : |t| < 1\}.$$

These disks are holomorphic at noncritical points of map (3.6) and foliate both spaces and the set Σ^0 .

The dilatations of the homotopy maps are estimated by

Proposition 3.2. [Kr3] *(a) Each homotopy map F_t of $F \in \Sigma$ admits k -quasiconformal extension to the complex sphere $\widehat{\mathbb{C}}$ with $k \leq |t|^2$. The bound $k(F_t) \leq |t|^2$ is sharp and occurs only for the maps*

$$F_{b_0, b_1}(z) = z + b_0 + b_1 z^{-1}, \quad |b_1| = 1, \quad (3.7)$$

whose homotopy maps

$$F_{b_0, b_1 t^2}(z) = z + b_0 t + b_1 t^2 z^{-1} \quad (3.8)$$

have the affine extensions $\widehat{F}_{b_0, b_1 t^2}(z) = z + b_0 t + b_1 t^2 \bar{z}$ onto Δ .

(b) If $F(z) = z + b_0 + b_m z^{-m} + b_{m+1} z^{-(m+1)} + \dots$ ($b_m \neq 0$) for some integer $m > 1$, then the minimal dilatation of extensions of F_t is estimated by $k(F_t) \leq |t|^{m+1}$; this bound also is sharp.

In the second case,

$$h_F(0) = h'_F(0) = \dots = h_F^{(m)}(0) = \mathbf{0}, \quad h_F^{(m+1)}(0) \neq \mathbf{0},$$

and due to [KK],

$$k(F_t) = \frac{m+1}{2} |b_m| |t|^{m+1} + O(t^{m+2}), \quad t \rightarrow 0. \quad (3.9)$$

This bound is sharp; it holds for the maps

$$F_{m,t}(z) = \frac{1}{\kappa_{m,t}(1/z)} = z \left(1 - \frac{t}{z^{m+1}}\right)^{2/(m+1)} = z - \frac{2t}{m+1} \frac{1}{z^m} + \dots, \quad |t| \leq 1, \quad (3.10)$$

whose extremal extension to \mathbb{C} has Beltrami coefficient $\mu_{F_{m,t}}(z) = t|z|^{m-1}/z^{m-1}$ for $|z| < 1$.

The Teichmüller geodesic (extremal) disks

$$\Delta(\psi) = \{\phi_{\bar{\mathbf{T}}}(t\mu_0) : t \in \Delta\}$$

in the spaces \mathbf{T} and \mathbf{T}_1 are generated by $F \in \Sigma^0$ having extremal extensions with Beltrami coefficients $\mu_t(z) = t|\psi(z)|/\psi(z)$, where ψ is a holomorphic integrable quadratic differential on Δ and $\Delta \setminus \{0\}$, respectively. Such an extension is unique (up to a constant factor of ψ) in their equivalence classes.

In particular, any homotopy function F_t has such an extremal extension. The Teichmüller disks foliate dense subsets in \mathbf{T} and \mathbf{T}_1 (and in Σ^0); cf. [GL], [St].

3.3. Two generalizations of Gaussian curvature and circularly symmetric metrics.

The proof of Theorem 2.1 involves also subharmonic conformal metrics $\lambda(t)|dt|$ on the disk (with $\lambda(t) \geq 0$) having the curvature at most -4 in a somewhat generalized sense. As is well-known, the Gaussian curvature of a C^2 -smooth metric $\lambda > 0$ is defined by

$$\kappa_\lambda = -\frac{\Delta \log \lambda}{\lambda^2},$$

where Δ means the Laplacian $4\partial\bar{\partial}$.

A metric $\lambda(t)|dt|$ in a domain D on \mathbb{C} (or on a Riemann surface) has curvature less than or equal to K **in the supporting sense** if for each $K' > K$ and each t_0 with $\lambda(t_0) > 0$, there is a C^2 -smooth **supporting metric** $\hat{\lambda}$ for λ at t_0 (i.e., such that $\hat{\lambda}(t_0) = \lambda(t_0)$ and $\hat{\lambda}(t) \leq \lambda(t)$ in a neighborhood of t_0) with $\kappa_{\hat{\lambda}}(t_0) \leq K'$, or equivalently,

$$\Delta \log \lambda \geq -K\lambda^2. \quad (3.11)$$

A metric λ has curvature at most K **in the potential sense** at z_0 if there is a disk U about t_0 in which the function

$$\log \lambda + K \text{Pot}_U(\lambda^2),$$

where Pot_U denotes the logarithmic potential

$$\text{Pot}_U h = \frac{1}{2\pi} \int_U h(\zeta) \log |\zeta - t| d\xi d\eta \quad (\zeta = \xi + i\eta),$$

is subharmonic. Since $\Delta \text{Pot}_U h = h$ (in the sense of distributions), one can replace U by any open subset $V \subset U$, because the function $\text{Pot}_U(\lambda^2) - \text{Pot}_V(\lambda^2)$ is harmonic on U . The inequality (3.11) holds for the generic subharmonic metrics also in the sense of distributions. Note also that the condition of having curvature at most $-K$ in the potential sense is invariant under conformal maps.

Due to [Ro2], a conformal metric of curvature at most K in the supporting sense has curvature at most K also in the potential sense.

The following lemma concerns the circularly symmetric (radial) metrics and is a slight improvement of the corresponding Royden's lemma [Ro2] to singular metrics with a prescribed singularity at the origin.

Lemma 3.3. [Kr7] *Let $\lambda(|t|)d|t|$ be a circularly symmetric subharmonic metric on Δ such that*

$$\lambda(r) = mcr^{m-1} + O(r^m) \quad \text{as } r \rightarrow 0 \quad \text{with } 0 < c \leq 1 \quad (m = 1, 2, \dots), \quad (3.12)$$

and this metric has curvature at most -4 in the potential sense. Then

$$\lambda(r) \geq \frac{mcr^{m-1}}{1 - c^2r^{2m}}.$$

Note that all metrics subject to (3.12) are dominated by $\lambda_m(t) = m|t|^{m-1}/(1 - |t|^{2m})$.

4. PROOF OF THEOREM 2.1

1⁰. One may assume that the degree d of J is even, replacing, if needed, this functional by its square J^2 .

Using the relations (1.2), we represent J as a polynomial functional on Σ , which takes the form

$$J(f) = \tilde{J}(F_f) = \tilde{J}(b_0, b_1, \dots, b_{2n-3}; F_f''(\zeta_1), \dots, F_f^{(m_1)}(\zeta_1); \dots, F_f''(\zeta_p), \dots, F_f^{(m_p)}(\zeta_p)), \quad (4.1)$$

where $b_0 = -a_2$ and $\zeta_j = 1/z_j$. As was mentioned above, the admissible values of b_0 for $F(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0$ with $F(0) = 0$ range over the closed domain $F(\overline{\Delta}) = \widehat{\mathbb{C}} \setminus F(\Delta^*)$.

The functional $\tilde{J}(F) = J(f)$ on $F = F_f \in \Sigma^0$ extends to a holomorphic functional \mathcal{J} on the fiber space $\mathcal{F}(\mathbf{T}) = \{(S_F, b_0)\}$ and thereby on the space \mathbf{T}_1 , letting for $\mu \in \mathbf{Belt}(\Delta)_1$ and $f^\mu \in S^0$ with $\tilde{\mu}(z) = \mu(1/z)z^2/\bar{z}^2$,

$$\mathcal{J}(S_{F^\mu}, b_0(F^\mu)) = J(f^\mu) \quad (b_0(F^\mu)) = -a_2(f^\mu). \quad (4.2)$$

We rescale \mathcal{J} by

$$\mathcal{J}^0(S_F, b_0) = \frac{\mathcal{J}(S_F, b_0)}{M(J)} \quad \text{with} \quad M(J) = \max_S |J(f)|$$

to have a holomorphic map of $\mathcal{F}(\mathbf{T})$ to the unit disk. For any fixed point $z_* \in \overline{D}$, one determines by (4.1) a holomorphic function

$$g_*(\varphi) = \mathcal{J}^0(-F(z_*), \{b(\varphi)\}, \{F^{(m_j)}(\zeta_j(\varphi))\}) : \mathbf{T} \rightarrow \Delta$$

where $\{b(\varphi)\}$ and $\{F^{(m_j)}(\zeta_j(\varphi))\}$ denote the collections (b_1, \dots, b_{2n-3}) and

$$(F''(\zeta_1), \dots, F^{(m_1)}(\zeta_1); \dots; F''(\zeta_p), \dots, F^{(m_p)}(\zeta_p)),$$

respectively, regarded as functions of the Schwarzians $\varphi = S_F \in \mathbf{T}$.

In view of the maximum principle, it suffices to use only the boundary points z_* . We select on the unit circle S^1 a dense subset

$$e = \{z_1, z_2, \dots, z_m, \dots\},$$

so that the corresponding sequence of holomorphic maps

$$g_m(\varphi) = \tilde{J}^0(-F(z_m), \{b(\varphi)\}; \{F^{(m_j)}(\zeta_j(\varphi))\}) : \mathbf{T} \rightarrow \Delta, \quad m = 1, 2, \dots \quad (4.3)$$

satisfies

$$\sup_m |g_m(S_F)| = \sup_{\mathbf{T}_1} |\mathcal{J}^0(S_F, a_2)| = \sup_{\Sigma^0} |\tilde{J}^0(F)| = \max_S |J(f)|/M(J). \quad (4.4)$$

2⁰. First suppose that there exists an extremal of $J(f)$ satisfying the assumption (2.5), and consider the functions $f \in S$ obeying this inequality. The set of the corresponding Schwarzians S_{F_f} is dense in \mathbf{T} , and their maps (3.6) satisfy

$$h_F(0) = h'_F(0) = \mathbf{0}, \quad h''_F(0) \neq \mathbf{0}.$$

We split every homotopy function F_t of $F = F_f$ by

$$F_t(z) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \dots = F_{b_0, b_1 t^2}(z) + h(z, t).$$

For sufficiently small $|t|$, the remainder h is estimated by $h(z, t) = O(t^3)$ uniformly in z for all $|z| \geq 1$. Then, by the well-known properties of Schwarzians, we have

$$S_{F_t}(z) = S_{F_{b_0, b_1 t^2}}(z) + \omega(z, t),$$

where the remainder ω is uniquely determined by the chain rule

$$S_{w_1 \circ w}(z) = (S_{w_1} \circ w)(w')^2(z) + S_w(z),$$

and is estimated in the norm of \mathbf{B} by

$$\|\omega(\cdot, t)\|_{\mathbf{B}} = O(t^3), \quad t \rightarrow 0; \quad (4.5)$$

this estimate is uniform for $|t| < t_0$ (cf., e.g. [Be1], [Kr1]). Hence, in view of holomorphy, every map (4.3) satisfies for small $|t|$,

$$g_m(S_{F_t}) = g_m(S_{F_{b_0, b_1 t^2}}) + O(t^{d+1}),$$

where the term $O(t^{d+1})$ is estimated uniformly for all n , and therefore,

$$\mathcal{J}(S_{F_t}, b_0 t) = \mathcal{J}(S_{F_{b_0, b_1 t^2}}, b_0) + O(t^{d+1}), \quad t \rightarrow 0. \quad (4.6)$$

Since $\mathcal{J}(S_{F_t}, b_0 t) = t^d \mathcal{J}(S_F, b_0)$ (in view of d -homogeneity of the functionals $\tilde{\mathcal{J}}$ and \mathcal{J}), we also have

$$t^d \mathcal{J}(S_F, b_0) = \mathcal{J}(S_{F_{b_0, b_1 t^2}}, b_0 t) + O(t^{d+1}), \quad t \rightarrow 0. \quad (4.7)$$

The values $\mathcal{J}(S_{F_{b_0, b_1 t^2}}, b_0 t)$ can be sharply estimated from above by Proposition 3.1. To this end, denote by s the canonical complex parameter on the Teichmüller disks in $\mathcal{F}(\mathbf{T})$ generated by admissible (that is, nonvanishing on Δ^*) functions

$$F_{b_0, s}(z) = z + b_0 + s z^{-1},$$

whose extremal extensions onto $\overline{\Delta}$ are the affine maps

$$z \mapsto z + b_0 + s \bar{z}.$$

All these disks cover the underlying Teichmüller disk $\Delta(S_{F_0, s})$ in the base space \mathbf{T} (note that $S_{F_0, s} = S_{F_{b_0, s}}$; the functions $F_{0, s}$ with $b_0 = 0$ are associated with points of \mathbf{T} in view of normalization $F_f(0) = 0$).

If $F_{b_0, s}$ is admissible only for $|s| < s_0 < 1$, one can reparametrize it using the parameter $\sigma = s/s_0$ which runs over the unit disk. Then, for each b_0 , the map

$$\sigma \mapsto (S_{F_{b_0, \sigma}}, b_0 \sigma), \quad \sigma \in \Delta,$$

is a complex geodesics in the space $\mathcal{F}(\mathbf{T})$ and

$$d_{\mathbf{T}_1}(\mathbf{0}, (S_{F_{b_0, s}}, b_0 s)) = d_{\mathbf{T}}(\mathbf{0}, S_{F_{b_0, s}}),$$

and similarly for the Carathéodory distances.

By (3.10), the parameters s and t are related near the origin by

$$s = b_1 t^2 + O(t^3) \quad (b_1 \neq 0), \quad (4.8)$$

so the restrictions of \mathcal{J}^0 and of g_m to the indicated Teichmüller disks both have at the origin zero of order $d/2$. We have

$$\mathcal{J}^0(S_{F_{b_0, s}}) = \beta_{d/2}(b_0) s^{d/2} + \beta_{d/2+1}(b_0) s^{d/2+1} + \dots,$$

and after estimating this map by Proposition 3.1 (applied to $\tilde{\mathbf{T}} = \mathbf{T}_1$ and $m = d/2$),

$$|\mathcal{J}^0(S_{F_{b_0,s}})| \leq |s|^{d/2} \frac{|s| + |\beta_{d/2}(b_0)|}{1 + |\beta_{d/2}(b_0)||s|}, \quad |s| < 1.$$

Replacing s by (4.8) and applying the relation (4.6), one obtains

$$|\mathcal{J}^0(S_{F_t}, b_0 t)| = |\mathcal{J}^0(S_{F_{b_0, b_1 t^2}}, b_0 t)| + O(t^{d+1}) \leq |\beta_{d/2}(b_0)| |b_1|^{d/2} |t|^d + O(t^{d+1}),$$

and after maximizing over admissible b_0 ,

$$\max_{b_0} |\mathcal{J}^0(S_{F_t}, b_0 t)| = \max_{b_0} |\mathcal{J}^0(S_{F_{b_0, b_1 t^2}}, b_0 t)| + O(t^{d+1}) \leq \max_{b_0} |\beta_{d/2}(b_0)| |b_1|^{d/2} |t|^d + O(t^{d+1}), \quad (4.9)$$

where all ratios $O(t^{d+1})/t^{d+1}$ remain uniformly bounded as $t \rightarrow 0$.

Now our goal is to show that the right-hand side of (4.9) yields simultaneously the lower asymptotic bound for $|\mathcal{J}^0(S_{F_t}, b_0 t)|$ for small $|t|$ (cf. [Kr7]), and therefore, the last inequality in (4.10) is reduced to an equality.

Lemma 4.1. *For any $F(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0$, we have*

$$\max_{b_0} |\mathcal{J}^0(S_{F_t}, b_0 t)| = \max_{b_0} |\mathcal{J}^0(F_{b_0, b_1 t^2})| + O(t^{d+1}) \geq \max_{b_0} |\beta_{d/2}(b_0)| |b_1|^{d/2} |t|^d + O(t^{d+1}). \quad (4.10)$$

again taking the maximum over admissible b_0 .

Proof. The homotopy disk of any function F_{b_0, b_1} is extremal and admits the rotational symmetry. Accordingly, $|\mathcal{J}(F_{b_0, b_1 t^2})|$ is circularly symmetric on the disk Δ . Define by (4.3) the corresponding functions

$$g_{m, b_1}(t) = \mathcal{J}^0(S_{F_{b_0, b_1 t^2}}, -F_{b_0, b_1 t^2}(z_m))$$

and conformal metrics

$$\lambda_{g_{m, b_1}}(t) = g_{m, b_1}^* \lambda_{\Delta}(t) = \frac{|g'_{m, b_1}(t)|}{1 - |g_{m, b_1}(t)|^2}$$

(whose Gaussian curvature equals -4 at noncritical points) and take the upper envelopes

$$\mathcal{J}^0(t) := \sup_m |g_{m, b_1}(t)|, \quad \lambda_{\mathcal{J}^0}(t) := \sup_m \lambda_{g_{m, b_1}}(t).$$

Both envelopes are circularly symmetric continuous and subharmonic on Δ . Note also that (cf. (4.4)),

$$\mathcal{J}^0(t) \leq |\mathcal{J}^0(S_{F_t}, b_0 t)| = \sup_m |g_{m, b_1}(S_{F_{b_0, b_1 t^2}})| = |\beta_{d/2}(b_0)| |b_1|^{d/2} |t|^d + O(t^{d+1}). \quad (4.11)$$

Since

$$\tanh^{-1} |g_{m, b_1}(r)| = \int_0^{|g_{m, b_1}(r)|} \frac{|dt|}{1 - |t|^2} = \int_0^r \lambda_{g_{m, b_1}}(t) |dt|,$$

we have

$$\tanh^{-1} |\mathcal{J}^0(r)| = \sup_m \int_0^r \lambda_{g_{m, b_1}}(t) |dt| = \int_0^r \sup_m \lambda_{g_{m, b_1}}(t) |dt| = \int_0^r \lambda_{\mathcal{J}^0}(t) dt. \quad (4.12)$$

The second equality in (4.12) is obtained by taking a monotone increasing subsequence of metrics

$$\lambda_1 = \lambda_{g_1, b_1}, \quad \lambda_2 = \max(\lambda_{g_1, b_1}, \lambda_{g_2, b_1}), \quad \lambda_3 = \max(\lambda_{g_1, b_1}, \lambda_{g_2, b_1}, \lambda_{g_3, b_1}), \quad \dots$$

so that

$$\lim_{p \rightarrow \infty} \lambda_p(t) = \sup_m \lambda_{g_m, b_1}(t).$$

Combining (4.12) with (4.4) and (4.7), one gets for small r ,

$$\tanh^{-1}[\mathcal{J}^0(S_{F_1}, b_0 r)] = \tanh^{-1}[\mathcal{J}^0(r)] + O(r^{d+1}) = \int_0^r \lambda_{\mathcal{J}^0}(t) dt + O(r^{d+1}). \quad (4.13)$$

Now observe that the metric $\lambda_{\mathcal{J}^0}$ has in a neighborhood of any $t_0 \in \Delta$ a supporting metric of curvature -4 , and therefore its curvature in Δ in the potential sense is at most -4 . Thus this metric can be estimated from below by Lemma 3.3 (with $m = d$) which implies the lower bound

$$\lambda_{\mathcal{J}^0}(r) \geq \frac{dCdr^{d-1}}{1 - C^2r^{2d}}, \quad r < 1, \quad (4.14)$$

where

$$C = \max_{b_0} |\beta_{d/2}(b_0)| |b_1|^{d/2}.$$

Integrating (4.14) over a small radial segment $[0, r]$, one obtains

$$\int_0^r \lambda_{\mathcal{J}^0}(t) dt \geq \tanh^{-1}(Cr^d) + O(r^{d+1}), \quad r \rightarrow 0,$$

which provides after substitution into (4.13) the desired estimate (4.11).

Comparison of the relations (4.7) and (4.9)-(4.11) yields

$$\begin{aligned} r^d |\mathcal{J}^0(S_F, b_0)| &= |\mathcal{J}^0(S_{F_r}, b_0 r)| = \max_{b_0} |\tilde{\mathcal{J}}^0(F_{b_0, b_1 r^2})| + O(r^{d+1}) \\ &= \max_{b_0} |\beta_{d/2}(b_0)| |b_1|^{d/2} r^d + O(r^{d+1}), \quad r \rightarrow 0, \end{aligned}$$

and letting $r \rightarrow 0$,

$$\begin{aligned} |\mathcal{J}^0(S_F, b_0)| &= \max_{b_0} |\tilde{\mathcal{J}}^0(F_{b_0, b_1})| = \max_{b_0} |\beta_{d/2}(b_0)| |b_1|^{d/2} \\ &= \max_{b_0} |\beta_{d/2}(b_0)| \left(\frac{|S_f(0)|}{6} \right)^{d/2} \leq 1. \end{aligned} \quad (4.15)$$

This estimate is established for all $f \in S$. Since for any extremal function $f_0(z) = z + \sum_{n=2}^{\infty} a_n^0 z^n$ of the functional J and its inversion $F_0(z) = F_{f_0}(z) = z + b_0^0 + b_1^0 z^{-1} + \dots$ obeying (2.3) must be

$$\frac{|\tilde{J}(S_{F_0}, -a_2^0)|}{M(J)} = |\mathcal{J}^0(S_{F_0}, -a_2^0)| = 1$$

and $|\beta_{d/2}(b_0)| \leq 1$, it follows from (4.15) that necessarily $\max_{b_0} |\beta_{d/2}(b_0)| = 1$ and

$$|b_1^0| = \frac{1}{6} |S_{f_0}(0)| = |(a_2^0)^2 - a_3^0| = 1.$$

As was mentioned, such equalities can only occur when f_0 either is the Koebe function κ_θ or it coincides with the odd function $\kappa_{2,\theta}$ defined by (1.7). In addition, the extremality of f_0 implies

$$|J(f_0)| = M(J) = \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\}.$$

3⁰. The functions $f \in S$ with $S_f(0) = 0$ omitted above can be approximated (in \mathbf{B} -norm) by f with $S_f(0) \neq 0$ by applying special quasiconformal deformations of the plane given by the following lemma from [Kr1, Ch. 4]. This lemma softens the strongest rigidity of conformal maps.

Lemma 4.2. *In a finitely connected domain $D \subset \widehat{\mathbb{C}}$, let there be selected a set E of positive two-dimensional Lebesgue measure and the distinct finite points z_1, \dots, z_n with assigned non-negative integers $\alpha_1, \dots, \alpha_n$, respectively, so that $\alpha_j = 0$ for $z_j \in E$. Then, for sufficiently small $\varepsilon > 0$ and $\varepsilon \in (0, \varepsilon_0)$, for any given system of numbers $\{w_{sj}\}$, $s = 0, 1, \dots, \alpha_j$, $j = 1, \dots, n$, such that $w_{0j} \in D$,*

$$|w_{0j} - z_j| \leq \varepsilon, \quad |w_{1j} - 1| \leq \varepsilon, \quad |w_{sj}| \leq \varepsilon \quad (s = 2, \dots, \alpha_j, \quad j = 1, \dots, n),$$

there exists a quasiconformal automorphism h_ε of the domain D , which is conformal on the set $D \setminus E$ and satisfies $h_\varepsilon^{(s)}(z_j) = w_{sj}$ for all $s = 0, 1, \dots, \alpha_j$ and $j = 1, \dots, n$, with dilatation $\|\mu_{h_\varepsilon}\|_\infty \leq M\varepsilon$. The constants ε_0 and M depend only on D , E and the vectors (z_1, \dots, z_n) , $(\alpha_1, \dots, \alpha_n)$.

If the boundary Γ of domain D is Jordan or belongs to the class $C^{l,\alpha}$, where $0 < \alpha < 1$ and $l \geq 1$, one can take $z_j \in \Gamma$ with $\alpha_j = 0$ or $\alpha_j \leq l$, respectively.

Now, let $f \in S^0$ have coefficients a_2 and a_3 related by $a_3 = a_2^2$, i.e., $b_1(f) := b_1(F_f) = 0$. Since $f(\Delta^*)$ is a domain, one can take there a set E of positive measure and construct by Lemma 4.2 for a sequence $\varepsilon_n \rightarrow 0$ such variations $h_n = h_{\varepsilon_n}$ of f that for each n ,

$$b_1(h_n \circ f) = b_1(f) + O(\varepsilon_n) \neq 0, \quad |J(h_n \circ f)| = |J(f)| + O(\varepsilon_n) > |J(f)|.$$

Since, by the previous step,

$$|J(h_n \circ f)| \leq \max\{|J(\kappa_\theta)|, |J(\kappa_{2,\theta})|\},$$

the same estimate will hold also for f .

4⁰. Finally, consider the case when J has no extremals f_0 satisfying (2.3), and hence any extremal inversion F_{f_0} is of the form

$$F(z) = z + b_0 + b_m z^{-m} + b_{m+1} z^{-(m+1)} + \dots \quad (b_m \neq 0; |z| > 1) \quad (4.16)$$

with $m > 1$. If $m+1$ does not divide $d = d(J)$, we consider the functional J^{m+1} , which is $d(m+1)$ -homogeneous; otherwise one can use J .

One can apply to J^{m+1} the above arguments, replacing $F_{0,b_1 t^2}$ by the corresponding function $F_{m,t}(z) + b_0$, where $F_{m,t}$ is given by (3.10) and b_0 is the same as in (4.16). Its Schwarzian relates to S_{F_t} by $S_{F_t} = S_{F_{m,t}} + O(t^{m+1})$ as $t \rightarrow 0$. Now

$$[J(S_{F_{m,t}}, b_0 t)/M(J)]^{m+1} = \beta_d(b_0) t^{d(m+1)} + \dots,$$

and after applying the asymptotic estimate (3.9), one obtains instead of (4.15) the bound

$$\left| \frac{\mathcal{J}(S_F, b_0)}{M(J)} \right|^{m+1} \leq \max_{b_0} |\beta_d(b_0)| \left(\frac{m+1}{2} |b_m| \right)^d \leq 1,$$

or

$$\left| \frac{\mathcal{J}(S_F, b_0)}{M(J)} \right| \leq \max_{b_0} |\beta_d(b_0)|^{1/(m+1)} \left(\frac{m+1}{2} |b_m| \right)^{d/(m+1)} \leq 1. \quad (4.17)$$

In the case of an extremal function $F_{f_0}(z) = z + b_0^0 + b_m^0 z^{-m} + \dots$ for $\tilde{J}(F)$, it must be $|\mathcal{J}(S_{F_{f_0}})/M(J)| = 1$, and (4.17) implies

$$\frac{m+1}{2} |b_m^0| = 1.$$

Since the functions (4.16) with $m > 1$ satisfy $b_1 = \dots = b_{m-1} = 0$, one can apply the well-known coefficient estimates of Golusin and Jenkins (see [Go, Ch. XI], [Je]) which provide in our case the bound

$$|b_m| \leq 2/(m+1) \quad (4.18)$$

with equality only for $F = F_{m,t}$ with $|t| = 1$ (up to translation $F_{m,t}(z) + c$). In this case, $M(J) = |J(\kappa_{m,\theta})|$.

We have established that any extremal function f_0 maximizing $|J(f)|$ must be of the form (2.1). The theorem is proved.

5. PROOF OF THEOREM 2.2

Note that from (1.3),

$$a_n^2 - a_{2n-1} = b_1 b_0^{2n-4} + \text{lower terms with respect to } b_0.$$

We have to show that for all $m > 1$,

$$|J_n(\kappa_{m,\theta})| < J_n(\kappa_0), \quad \kappa_0 = z/(1-z)^2. \quad (5.1)$$

Then Theorem 2.1 implies that only the Koebe function is extremal for Zalcman's functional.

This inequality is trivial for $m = 2$, because the series (1.7) yields

$$|J_n(\kappa_{2,\theta})| \leq 2 < J_n(\kappa_0).$$

For $m \geq 3$, we apply a result of [Kr5] solving the coefficient problem for univalent functions with quasiconformal extensions having small dilatations. Denote by $S_\infty(k)$ the subclass of S^0 consisting of the functions $f \in S$ having k' -quasiconformal extensions \hat{f} to $\hat{\mathbb{C}}$ ($k' \leq k$) which satisfy $\hat{f}(\infty) = \infty$, and let

$$f_{1,t}(z) = \frac{z}{(1-ktz)^2}, \quad |z| < 1, \quad |t| = 1.$$

Proposition 5.1. [Kr5] *For all $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\infty(k)$ and all $k \leq 1/(n^2+1)$, we have the sharp bound*

$$|a_n| \leq 2k/(n-1), \quad (5.2)$$

with equality only for the functions

$$f_{n-1,t}(z) = f_{1,t}(z^{n-1})^{1/(n-1)} = z + \frac{2kt}{n-1} z^n + \dots, \quad n = 3, 4, \dots \quad (5.3)$$

Note that every function (5.3) admits a quasiconformal extension $\widehat{f}_{n-1,t}$ onto Δ^* with Beltrami coefficient $\mu_n(z) = t|z|^{n+1}/z^{n+1}$ and $\widehat{f}_{n-1,t}(\infty) = \infty$. Accordingly, $\widehat{F}_{n-1,t}(z) = 1/\widehat{f}_{n-1,t}(1/z) \in \Sigma^0$ admits a quasiconformal extension onto the unit disk with $\widehat{F}_{n-1,t}(0) = 0$ and $\mu_{\widehat{F}_{n-1,t}}(z) = t|z|^{n-1}/z^{n-1}$ for $|z| < 1$. Another essential point is that for any function

$$F_{n-1}(z) := \widehat{F}_{n-1,1}(z) = 1/\kappa_0(1/z^{n-1})^{1/(n-1)},$$

its homotopy disk $\{S_{F_{n-1}}\}$ in \mathbf{T} is Teichmüller geodesic. Together with estimate (5.2), this implies that for any $m > 2$ and small $r > 0$,

$$|J_n(\kappa_{m,r})| < r(n-1)^2;$$

thus

$$|J_n(\kappa_{m,\theta})| < (n-1)^2 = J_n(\kappa_0),$$

completing the proof of Theorem 2.2.

Remarks.

1. The above arguments work well also in the case of functionals obtained by suitable perturbation of $J_n(f)$. For example, one can take

$$J(f) = a_n^2 - a_{2n-1} + P(a_3, \dots, a_{2n-2}), \quad (5.4)$$

where P is a homogeneous polynomial of degree $2n-2$,

$$P(a_3, \dots, a_{2n-2}) = \sum_{|k|=2n-2} c_{k_3, \dots, k_n} a_3^{k_2} \dots a_n^{k_{2n-2}},$$

and $|k| := k_3 + \dots + k_{2n-2}$, $a_j = a_j(f)$, assuming that this polynomial has nonnegative coefficients and satisfies

$$\max_S |P(a_3, \dots, a_{2n-2})| < \frac{(n-1)^2}{2}.$$

For any such functional, only the Koebe function is extremal.

2. One can simplify the above proof applying instead of Proposition 5.1 a weaker estimate $|a_n| \leq 2k/(n-1)^2 + O(k^2)$ following from the variational formula for $f \in S(k)$.

3. The evaluation of the coefficient functionals $J(F) = J(b_{m_1}(F), \dots, b_{m_p}(F))$ on the class Σ is somewhat different. In view of normalization, this case relates to the space the universal Teichmüller space \mathbf{T} (instead of \mathbf{T}_1 for the class S).

The arguments exploited in the last step of the proof of Theorem 2.1 do not work for the generic functionals on Σ , for example, if $J(F) = b_m + \xi b_1$ with small ξ , because there are functions $F \in \Sigma$ for which the inequality (4.18) does not hold.

6. SOME NEW DISTORTION THEOREMS FOR HIGHER COEFFICIENTS

As was mentioned, Theorem 3.1 provides various new distortion estimates. For example, we obtain the following generalizations of the inequality $|a_2^2 - a_3| \leq 1$ to higher coefficients.

Theorem 6.1. *For all $f \in S$ and integers $n > 3$ and $p \geq 1$,*

$$|a_n^p - a_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p.$$

This bound is sharp, and the equality only occurs for the Koebe function κ_θ .

Proof. Since $b_0 = -a_2$, the relation (3.1) yields

$$I_n(f) := a_n - a_2^{n-1} = (n-2)(-1)^{n-1}b_1b_0^{n-3} + \text{lower terms with respect to } b_0.$$

This functional and $I_n^p(f) = |a_n^p - a_2^{p(n-1)}|$ satisfy the assumptions of Theorem 2.1. The same arguments as in the proof of Theorem 2.2 imply

$$|I_n^p(\kappa_{m,\theta})| < |I_n^p(\kappa_\theta)| \quad \text{for all } m \geq 2,$$

completing the proof.

In the same way, one obtains

Theorem 6.2. *For all $f \in S$ and integers $n > 2$ and $p \geq 1$,*

$$|a_{n+1}^p - a_2^p a_n^p| \leq 2^p n^p - (n+1)^p,$$

with equality only for $f = \kappa_\theta$.

7. ASYMPTOTIC THEOREMS

Another consequence of Theorem 2.1 concerns the asymptotic rate of growth of generic homogeneous functionals on individual functions with quasiconformal extension (which are dense in S). We present here somewhat restricted results.

Theorem 7.1. *Let $\{J_j(f)\}_1^\infty$ be a sequence of uniformly bounded homogeneous holomorphic (not necessarily distinct) functionals on S of degrees $d_j = d(J_j)$ satisfying the assumptions of Theorem 2.1 and such that*

$$|J_j(\kappa_{m,\theta})| \leq |J_j(\kappa_0)| \quad \text{for all } m > 1. \quad (7.1)$$

Then for any $f \in S^0$,

$$\limsup_{j \rightarrow \infty} \left| \frac{J_j(f)}{J_j(\kappa_0)} \right| = v(f) < 1. \quad (7.2)$$

A similar result holds for the upper envelop $\sup_\alpha |J_\alpha(f)|$ of a family of uniformly bounded homogeneous holomorphic functionals on S .

Proof. It follows from (7.1) and Theorem 2.1 that only the Koebe function κ_θ is extremal for each J_j . We lift $J_j(f)$ to holomorphic functionals $\widehat{J}_j(\varphi)$ on the space \mathbf{T}_1 (taking again $\varphi = (S_f, -a_2(f))$) and consider the ratios

$$v(\varphi) = \limsup_{j \rightarrow \infty} \left| \frac{\widehat{J}_j(\varphi)}{\widehat{J}_j(\varphi_0)} \right|$$

where $\varphi_0 = (S_{\kappa_0}, -2)$. The function $v(\varphi)$ is well defined on this space and $v(\varphi) \leq 1$. Its upper semicontinued regularization $v^*(\varphi) = \limsup_{\varphi' \rightarrow \varphi} v(\varphi') \leq 1$ is plurisubharmonic on \mathbf{T}_1 , hence by

the maximum principle it cannot attain the value 1 inside \mathbf{T}_1 ; otherwise this function must be identically equal to 1. But, for example, $v^*(\mathbf{0}) = v(\mathbf{0}) = 0$, since near the origin by Schwarz's lemma,

$$\left| \frac{\widehat{J}_j(\varphi)}{\widehat{J}_j(\varphi_0)} \right| \leq \text{const } \|\varphi\|$$

for all p . Therefore, $v(\varphi) \leq v^*(\varphi) < 1$, which completes the proof of (7.2).

In the case of Zalcman's functional this yields that for any $f \in S^0$,

$$v(f) = \limsup_{n \rightarrow \infty} \frac{|a_n^2 - a_{2n-1}|}{(n-1)^2} < 1$$

(cf. [Ha], [EV]). On the other hand, the bound (1.5) implies that $v(f) \leq 1$ for any $f \in S$. Similar estimates hold for perturbations of this functional via (5.4).

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